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First integrals of the equations of motion of a symmetric gyrostat on a perfectly rough plane☆

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Abstract

The problem of the motion of a dynamically symmetric gyrostat without slipping on a fixed horizontal plane is investigated. When the surface of the gyrostat and the distribution of the masses in it satisfy a certain condition, supplementing and developing the results obtained by Mushtari [Mushtari KhM. The rolling of a heavy solid of revolution on a fixed horizontal plane. *Mat Sbornik* 1932; **39**(1, 2):105–26], an explicit form of two first integrals of the equations of motion of the gyrostat, in addition to the energy integral, is presented.

Chaplygin,[2] when establishing the integrability of the problem of the motion of a dynamically symmetric solid of revolution on a perfectly rough plane, pointed out that the problem remains integrable when a uniformly rotating dynamically and geometrically symmetric rotor, whose axis of rotation coincides with the axis of symmetry of the body, is added to the body. The equations of motion of a gyrostat of similar form, in addition to an energy integral, allow of two first integrals, that are linear in the generalized velocities. However, an explicit form of these integrals is only known in the case when the gyrostat is bounded by a spherical surface.[2,3] When the gyrostat is a heavy disc, carrying the rotor, the axis of rotation of which is perpendicular to the plane of the disc and passes through its centre of mass, the integrals, linear in the velocities, are expressed using hypergeometric series.[2,4] For additional restrictions, imposed on the moments of inertia of the body and on the value of the gyrostatic moment of the rotor, a further two cases were obtained in Ref. 1 when all the first integrals of this problem can be expressed in explicit form, namely (1) the surface of the body, carrying the rotor, is formed by rotating the arc of the parabola about the axis passing through its focus, and (2) the body is a paraboloid of revolution. For gyrostats, whose surface has a different shape, an explicit form of the first integrals, linear in the velocities, is unknown. Below we attempt to obtain some new cases when all the integrals of the problem can be expressed explicitly.

1. Formulation of the problem and the equations of motion

Suppose a rigid body, symmetrical in shape and mass distribution about a certain axis, is at rest at a point M on a fixed horizontal plane Oxy. We connect a rotor to the body, the axis of the rotor coinciding with the axis of symmetry of the body. The body with the rotor represents a dynamically symmetric gyrostat.[5] We will use the following notation:

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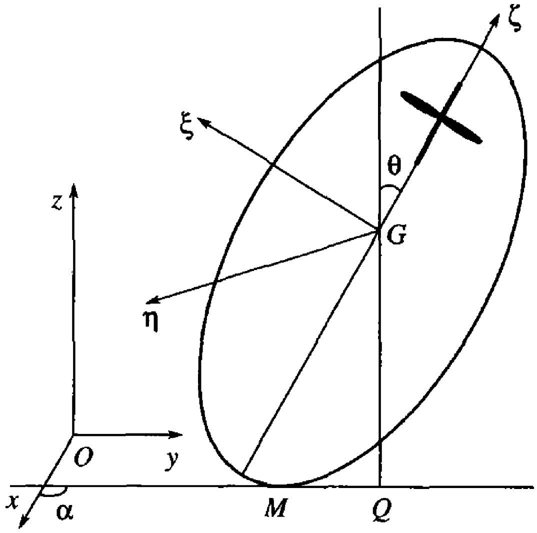

Fig. 1.

θ is the angle between the axis of symmetry of the body and the vertical, β is the angle between the meridian $M\zeta$ of the body and some fixed meridional plane, and α is the angle between the horizontal tangent MQ of the meridian $M\zeta$ and the Ox axis. The position of the body will be completely defined by the angles α, β and θ, and the x and y coordinates of the point M.

In addition, we will introduce a system of coordinates $G\xi\eta\zeta$ with origin at the centre of mass G of the gyrostat, which moves both in space and in the body so that the $G\zeta$ axis coincides with the axis of symmetry of the body, the $G\xi$ axis lies all the time in the plane of the vertical meridian, while the $G\eta$ axis is perpendicular to this plane (see the Fig. 1). Suppose the velocity vector $\mathbf{v}$ of the centre of mass G, the angular velocity vector $\boldsymbol{\omega}$ of the body, the angular velocity vector $\boldsymbol{\Omega}$ of the trihedron $G\xi\eta\zeta$ and the reaction of the plane $\mathbf{R}$ are specified in the system of coordinates $G\xi\eta\zeta$ by the components υ_ξ, υ_η, υ_ζ; p, q, r; Ω_ξ, Ω_η, Ω_r and R_ξ, R_η, R_ζ respectively. Suppose m is the mass of the gyrostat, A_1 is the moment of inertia of the gyrostat about the $G\xi$ and $G\eta$ axes, and A_3 is the moment of inertia of the body about the $G\zeta$ axis. We will denote the gyrostatic moment of the rotor about its axis of rotation by s; ignoring friction in the bearings of the rotor axis, the quantity s can be assumed to be constant.

Note[6] that the distance GQ from the centre of gravity to the Oxy plane will be a function of the angle θ i.e. $GQ = f(\theta)$. The coordinates ξ, η, ζ of the point M where the body touches the plane in the system of coordinates $G\xi\eta\zeta$ will also be functions of only the angle θ, where $\eta = 0$, and

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta \tag{1.1}$$

Since the $G\zeta$ axis is fixed in the body, we have $\Omega_\xi = p$, $\Omega_\eta = q$. The $G\xi\zeta$ plane will all the time be vertical, and hence $\Omega_\zeta - \Omega_\xi \operatorname{ctg}\theta = 0$. The velocity of the point of contact is equal to zero, and consequently

$$\upsilon_\xi + q\zeta = 0, \quad \upsilon_\eta + r\xi - p\zeta = 0, \quad \upsilon_\zeta - q\xi = 0$$

The theorem of the motion of the centre of mass as a projection onto the $G\eta$ axis and the theorem of the change in the angular momentum for the $G\xi$ and $G\zeta$ axes, after simple transformations, give

$$\begin{gathered} \frac{d(p\zeta - r\xi)}{dt} - pq(\zeta\operatorname{ctg}\theta + \xi) = \frac{R_\eta}{m} \\ A_1\frac{dp}{dt} + (A_3r + s - A_1p\operatorname{ctg}\theta)q = -\zeta R_\eta, \quad A_3\frac{dr}{dt} = \xi R_\eta \end{gathered} \tag{1.2}$$

Omitting henceforth the special case when $\theta = \text{const}$ and bearing in mind that $q = -d\theta/dt$, by eliminating R_η from system (1.2) we obtain

$$A_1\frac{dp}{d\theta} + A_3\frac{\zeta}{\xi}\frac{dr}{d\theta} = -A_1 p\,\text{ctg}\,\theta + A_3 r + s$$
$$\zeta\frac{dp}{d\theta} - \frac{A_3 + m\xi^2}{m\xi}\frac{dr}{d\theta} = -(\zeta\,\text{ctg}\,\theta + \xi + \zeta')p + \xi' r \tag{1.3}$$

Hence, from system (1.3) we can determine two first integrals of the equations of motion of the gyrostat that are linear in p and r. At present an explicit form of these integrals can only be obtained in some special cases.[1–4] In this paper we point out some new cases when, in the problem of the motion of a dynamically symmetric gyrostat, one can obtain first integrals in explicit form that are linear in p and r.

2. Derivation of the first integrals

The system of equations (1.3) can be represented in the form

$$\frac{d\tau}{d\theta} = h(\theta)r + u_1(\theta)s, \quad \frac{dr}{d\theta} = z(\theta)\tau + u_2(\theta)s \tag{2.1}$$

Here we have introduced the following notation

$$\tau = m\sqrt{\Delta}[A_1 p\sin\theta + Br]$$

$$h(\theta) = m\sqrt{\Delta}\left(A_3\sin\theta + \frac{dB}{d\theta}\right), \quad z(\theta) = \frac{\xi(\xi + \zeta')}{\Delta^{3/2}\sin\theta}$$

$$u_1(\theta) = m\sqrt{\Delta}\sin\theta\left[1 - \frac{m\zeta(A_1\xi\xi' + A_3\zeta\zeta')}{\Delta(\xi + \zeta')}\right], \quad u_2(\theta) = \frac{m\xi\zeta}{\Delta}$$

$$\Delta = A_1A_3 + A_1m\xi^2 + A_3m\zeta^2, \quad B = \frac{(A_3\zeta - A_1\xi')\sin\theta}{\xi + \zeta'}$$

We will assume that the surface of the gyrostat and the distribution of the masses in it are such that the following condition is satisfied

$$B = A_3(\cos\theta + \sigma) \tag{2.2}$$

where σ is an arbitrary constant.

Condition (2.2) was written for the first time in Ref. 1 when investigating the motion of a heavy dynamically symmetric solid of revolution (without a rotor). It was shown there that when this condition is satisfied the equations of motion of a solid of revolution (having the form of Eq. (1.3) in which we put $s = 0$) allow of a first integral $r = r_0 = \text{const}$. In this case, when condition (2.2) is satisfied we have $h(\theta) \equiv 0$, and it follows from the first equation of system (2.1) that the following quantity is preserved

$$\tau - s\varphi_1(\theta) = \tau_0 = \text{const}; \quad \varphi_1(\theta) = \int_{\theta_0}^{\theta} u_1(t)dt \tag{2.3}$$

After some simplification, this integral can be represented in the form

$$\sqrt{\Delta}\left[A_1 p\sin\theta + A_3 r(\cos\theta + \sigma) + \frac{\zeta\sin\theta}{\xi + \zeta'}s\right] - s\int_{\theta_0}^{\theta}\sqrt{\Delta}\left[\frac{d}{dt}\left(\frac{\zeta\sin t}{\xi + \zeta'}\right) + \sin t\right]dt = c_1 \tag{2.4}$$

After finding integral (2.4) it is easy to obtain the other first integral in explicit form. Using relation (2.3), we obtain

$$\tau = \tau_0 + s\varphi_1(\theta)$$

Substituting the expression obtained for τ into the second equation of system (2.1), we reduce it to the form

$$dr/d\theta = z(\theta)\tau_0 + (u_2(\theta) + z(\theta)\varphi_1(\theta))s$$

Consequently, Eq. (2.1), in addition to integral (2.4), also allow of the following integral

$$r - mc_1\psi_1(\theta) - s\psi_2(\theta) = c_2$$

$$\psi_1(\theta) = \int_{\theta_0}^{\theta} \frac{\xi(\xi + \zeta')}{\Delta^{3/2}\sin t}dt, \quad \psi_2(\theta) = \int_{\theta_0}^{\theta} [u_2(t) + z(t)\varphi_1(t)]dt \tag{2.5}$$

Hence, when condition (2.2) is satisfied the equations of motion of a dynamically symmetric gyrostat on a perfectly rough plane allow of first integrals (2.4) and (2.5). We will now investigate what shape the surface of the gyrostat should have in order for it to satisfy condition (2.2).

3. Determination of the shape of the gyrostat surface

Consider condition (2.2) and assume initially that $\sigma = 0$. Substituting expressions (1.1) for ξ, ζ and their derivatives into condition (2.2) and introducing the dimensionless parameter $k = A_3/A_1$, we find that the function $f(\theta)$, which defines the shape of the gyrostat surface, must satisfy the equation

$$(k-1)f''\sin\theta\cos\theta - kf' + (k-1)f\sin\theta\cos\theta = 0 \tag{3.1}$$

It is easy to show that Eq. (3.1) has two particular solutions

$$1)\ k = \frac{2}{3}, \quad f(\theta) = \frac{\lambda}{\sin\theta}, \quad 2)\ k = 2, \quad f(\theta) = \frac{\lambda}{\cos\theta} \tag{3.2}$$

which were pointed by Mushtari for the first time in Ref. 1. We will investigate what other solutions this equation has. We will put

$$f(\theta) = g(\theta)/\cos\theta$$

Then, we can write Eq. (3.1) for the function $g(\theta)$ as follows:

$$g'' + 2\left(\frac{\kappa}{\sin\theta\cos\theta} - \frac{\cos\theta}{\sin\theta}\right)g' + \frac{2\kappa}{\cos^2\theta}g = 0; \quad \kappa = \frac{k/2 - 1}{k - 1} \tag{3.3}$$

Making the change of independent variable in Eq. (3.3) in accordance with the formula

$$w = 1/\cos^2\theta$$

we reduce it to the Gauss hypergeometric equation[7]

$$w(1-w)\frac{d^2g}{dw^2} + \left[2 - \left(\frac{3}{2} + \kappa\right)w\right]\frac{dg}{dw} - \frac{\kappa}{2}g = 0 \tag{3.4}$$

Hence, Eq. (3.1) has the non-trivial particular solution

$$f_0(\theta) = \frac{1}{\cos\theta}F\left(\frac{1}{2}, \kappa, 2; \frac{1}{\cos^2\theta}\right)$$

and the general solution of Eq. (3.1) can be written as follows:

$$f(\theta) = f_0(\theta)\left(\lambda + \mu\int\frac{(\mathrm{tg}\theta)^{k/(k-1)}}{f_0^2(\theta)}d\theta\right) \tag{3.5}$$

Here $F = F(a, b, c; z)$ is the Gauss hypergeometric function.[7] We will show how one can obtain particular solutions (3.2) from the general solution (3.5). To do this we will make two assumptions. Suppose $\mu = 0$ in formula (3.5). Then the function $f(\theta)$ is given by the formula

$$F(\theta) = \frac{\lambda}{\cos\theta}F\left(\frac{1}{2}, \kappa, 2; \frac{1}{\cos^2\theta}\right)$$

We will now assume that the function F in this expression can be represented by a finite sum, i.e. one of its parameters is equal to a negative integer or zero.[7] In the case considered this means that $\kappa = -N$, where N is a natural number or zero. Hence we obtain

$$k = 2(N+1)/(2N+1) \tag{3.6}$$

Using the formula[7]

$$F(a, b, c; z) = (1-z)^{-a}F(a, c-b, c; z(z-1)^{-1})$$

we can also conclude that the hypergeometric series will be a finite sum, if the number $c - b$ is equal to a negative integer. In the case considered this condition gives $2 - \kappa = -N$, whence we find

$$k = 2(N+1)/(2N+3) \tag{3.7}$$

When $N = 0$, from Eq. (3.6) we obtain $k = 2$, and it follows from Eq. (3.7) that $k = 2/3$. Consequently, the particular solutions (3.2), given for the first time by Mushtari in Ref. 1, are obtained from the general solution (3.5) when $\mu = 0$ and $N = 0$. In the general case the set of functions which satisfy Eq. (3.1) is defined by formula (3.5) and has the power of a continuum.

We will now assume that the constant $\sigma \neq 0$ in condition (2.2). In this case the equation for determining the function $f(\theta)$ will have the form

$$\sin\theta((k-1)\cos\theta + k\sigma)f'' - k(1+\sigma\cos\theta)f' + (k-1)\sin\theta\cos\theta f = 0 \tag{3.8}$$

Making the replacement of the independent variable in (3.8) in accordance with the formula

$$w = \cos^2(\theta/2)$$

we reduce it to the equation

$$\begin{aligned} &w(w-1)(w-a_1)\frac{d^2f}{dw^2} + [(\alpha_1+\beta_1+1)w^2 - \\ &-[\alpha_1+\beta_1+1+a_1(\gamma_1+\delta_1)-\delta_1]w + a_1\gamma_1]\frac{df}{dw} + (\alpha_1\beta_1 w - q_1)f = 0 \end{aligned} \tag{3.9}$$

known as Heun's equation.[8,9] In the case considered here

$$a_1 = -\frac{1}{4\gamma_1}, \quad \alpha_1 = -\beta_1 = 1, \quad q_1 = -\frac{1}{2}, \quad \gamma_1 = \frac{1}{2(1+k(\sigma-1))}, \quad \delta_1 = \frac{1}{2(1-k(\sigma+1))}$$

When $|a_1| \geq 1$ and $\gamma_1 \neq 0, -1, -2, -3, \ldots$, the solution of Eq. (3.9) can be represented in the form of a series

$$F(a_1, q_1, \alpha_1, \beta_1, \gamma_1, \delta_1; w) = \sum_{n=0}^{\infty} c_n w^n \tag{3.10}$$

the coefficients of which are defined by the recurrence formulae

$$c_1 = 1, \quad a_1\gamma_1 c_1 = q_1$$

$$a_1(n+1)(\gamma_1+n)c_{n+1} = \left[a_1(\gamma_1+\delta_1+n-1)+\alpha_1+\beta_1-\delta_1+n+\frac{q_1}{n}\right]nc_n -$$
$$-[(n-1)(n-2)+(n-1)(\alpha_1+\beta_1+1)+\alpha_1\beta_1]c_{n-1}$$

This series necessarily converges when $|w| \leq 1$. Note, however, that Heun's equation (3.9) has been investigated considerably less than the hypergeometric equation. For example, unlike the hypergeometric function, there is no integral representation of Heun's function (3.10). Little is also known about to what functions Heun's function reduces for particular values of its parameters.

Hence, when $\sigma = 0$, the body surface, that satisfies condition (2.2), is determined using the hypergeometric function (see formula (3.5)), and when $\sigma \neq 0$ it is determined using Heun's function (3.10).

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